Attenuation, dispersion and finite propagation speed in viscoelastic media

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Abstract

It is shown that the dispersion and attenuation functions in a linear viscoelastic medium with a positive relaxation spectrum have a sublinear growth rate at very high frequencies. A local dispersion relation in parametric form is found. The exact limit between attenuation growth rates compatible and incompatible with finite propagation speed is found.

Incompatibility of superlinear frequency dependence of attenuation with finite speed of propagation and with the assumption of positive relaxation spectrum is demonstrated.

List of symbols.

$\tilde{f}(p)$	$\int_0^\infty f(t)\,\mathrm{e}^{-pt}\,\mathrm{d}t$	Laplace transform;
$\theta(t)$		unit step function;
t_+^α	$\theta(t)\|t\|^\alpha$	homogeneous distribution.
$\mathrm{D}_\mathrm{C}^\alpha$	$\mathrm{I}^{n-\alpha}\,\mathrm{D}^n$	Caputo fractional derivative

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1 Introduction.

A very elegant theory linking attenuation and dispersion is presented for viscoelastic media with positive relaxation spectrum. All the attenuation and dispersion functions compatible with the theory are represented by simple integral representations. The attenuation function and the dispersion function are both expressed as transforms of a positive measure (the dispersion-attenuation measure). The measure is arbitrary except for a very mild growth condition. These expressions can be considered as a dispersion relation in parametric form.

In contrast the acoustic Kramers-Kronig dispersion [29] relations are non-local. They express the dispersion function in terms of the attenuation function or conversely. This presupposes that one of these functions (usually the dispersion function) is very accurately known and consistent with the basic assumptions of the theory. On the other hand, substituting any positive measure in the parametric dispersion relation yields an admissible (compatible with the theory) dispersion and attenuation. The origin of the Kramers-Kronig dispersion relations is unclear. In electromagnetic theory they follow from the causality of the time-domain kernel representing the dielectric constant in the dispersive case. In acoustics they follow from an ad hoc assumption about the analytic properties of the wave number. It is namely assumed that the wave number is the Fourier transform of a causal function or distribution. The physical meaning of the causal function or distribution is unclear hence the justification of the acoustic Kramers-Kronig dispersion relation is missing. Various inequalities imposed on the complex continuation of the wave number cannot be expressed in terms of a constitutive assumption. Hence the acoustic Kramers-Kronig dispersion relation are an ad hoc addition to the constitutive equation, often incompatible with it.

It will be also shown that in media with non-negative relaxation spectrum the frequency dependence of the attenuation function in the high frequency range is sublinear. In the case of power law attenuation the attenuation and dispersion are proportional to a power of frequency $|\omega|^\alpha$. Numerous experiments in acoustics indicate that the power-attenuation law accurately represents the frequency dependence of dispersion and attenuation over several decades of frequency. The theory based on positive relaxation spectrum implies that $0 \leq \alpha < 1$. This seems to contradict the experimental ultrasound investigations of numerous materials which point to higher values of the exponent in the power law attenuation. For example in ultrasound investigations of soft tissues the exponent varies between 1 and 1.5, while in some viscoelastic fluids such as castor oil it lies between 1.5 and 2. We shall refer to this case as superlinear frequency dependence. Typical values of the power-law exponent in medical applications using ultrasound transducers are $\alpha = 1.3$ in bovine liver for 1–100 MHz, $\alpha = 1 - 2$ in human myocardium and other bio-tissues [27]. $\alpha = 1.66$ in castor oil at ca 250 MHz. Values in the range 1–2 are observed at lower frequencies in aromatic polyurethanes [11]. Nearly linear frequency dependence of attenuation is well documented in seismology [10]. Approximately linear frequency dependence of attenuation has been observed in geological materials in the range 140 Hz to

2.2 MHz.

Several papers have been devoted to a theoretical underpinning of the superlinear dispersion-attenuation models [25, 26, 6, 28, 7, 16]. In order to resolve some problems Chen and Holm [5] suggested to add a fractional Laplacian of order $0 < y < 2$ of the velocity field to the usual Laplacian of the displacement field in the equations of motion. Their paper still leads to an unbounded sound speed for $y > 1$ and adds a new problem: the equation of motion does not have the form of a viscoelastic equation of motion.

Sublinear power law attenuation ($\alpha < 1$) has also been reported in experimental investigations [21]. Non-power law attenuation laws are usually derived from the constitutive laws. Investigations of creep and relaxation in viscoelastic materials always support the assumption of positive relaxation spectrum (e.g. [8, 9] for creep in metals, [17] for the upper mantle with $\alpha = 1/3$) and therefore models derived from constitutive relations exhibit sublinear attenuation and dispersion at high frequencies. In particular this applies to the Cole-Cole, Havriliak-Negami and Cole-Davidson and Kohlrausch-Williams-Watts relaxation laws commonly applied in phenomenological rock mechanics, polymer rheology [1], bio-tissue mechanics (e.g. for bone collagen [22]) as well as for ionic glasses [4].

Another abnormal feature of wave propagation in media with superlinear power laws (and more generally in media with superlinear asymptotic frequency dependence) is appearance of precursors. The precursors extend to infinity and thus the speed of propagation of disturbances is infinite. Finite speed of wave propagation requires that in the high-frequency range the exponent of the power law does not exceed 1.

It is a very challenging problem how to explain the incompatibility between the theory and experiment in the superlinear case. It seems likely that the attenuation observed at ultrasound frequencies significantly differs from the asymptotic behavior of attenuation at the frequency tending to infinity.

One might surmise that the frequency range of ultrasound measurements is still far below the asymptotic high frequency range in which different mechanisms are at play. This suggests studying models of attenuation with a slowly varying power-law exponent.

2   Constitutive assumptions and basic definitions.

In viscoelasticity the relaxation modulus G, defined by the constitutive stress-strain relation

$$\sigma(t) = \int_0^t G(t - s)\, \dot{e}(s)\, \mathrm{d}s \tag{1}$$

is assumed to have positive relaxation spectral density h. The latter statement means that for every $t > 0$

$$G(t) = \int_0^\infty \mathrm{e}^{-tr}\, h(r)\, \mathrm{d}\ln(r), \qquad t > 0 \tag{2}$$

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where $r = 1/\tau$ is the inverse of the relaxation time and $h(r) \geq 0$. Eq. (2) represents a superposition of a continuum of Debye elements. For mathematical convenience eq. (2) will be replaced by a more general equation

$$G(t) = \int_{[0,\infty\,[} e^{-tr}\,\mu(dr), \qquad t > 0 \tag{3}$$

where μ is a positive measure: $\mu([a, b]) \geq 0$ for every interval $[a, b]$ of the positive real axis. As indicated in the subscript of the integral sign the range of integration is the set of reals satisfying the inequality $0 \leq r < \infty$. In general the measure $\mu(\{0\})$ of the one-point set $\{0\}$ is finite and equal to the equilibrium modulus $G_\infty := \lim_{t\to\infty} G(t)$. An additional assumption

$$\int_{[0,\infty[} \frac{\mu(dr)}{1 + r} < \infty \tag{4}$$

ensures that G is integrable over $[0, 1]$, [13]. The relaxation modulus assumes a finite value $G(0) = M$ at 0 if the measure μ has a finite mass M.

The right-hand side of eq. (3) can be replaced by a Stieltjes integral with respect to the function $g(r) = \mu([0, r])$:

$$G(t) = \int_{[0,\infty\,[} e^{-tr}\,dg(r) \tag{5}$$

The function g is non-decreasing and right-continuous: $g(r) = \mu([0, r]) = \lim_{\varepsilon\to0+} \mu([0, r + \varepsilon])$, and $g(r) = 0$ for $r < 0$.

In contrast to eq. (2) the integral representations (3) and (5) include as special cases finite spectra of relaxation times corresponding to superpositions of a finite number of Debye elements

$$G(t) = \sum_{n=1}^{N} c_n\,e^{-r_n t}, \qquad c_n > 0, r_n \geq 0 \quad \text{for } n = 1, \ldots, N \tag{6}$$

(Prony sums), infinite discrete spectra (corresponding to Dirichlet series) as well as discrete spectra embedded in continuous spectra. However the main advantage of (3) over (2) is the availability of very powerful mathematical theory which ensures logical equivalence of certain statements about material response functions.

In particular a function satisfying (3) is completely monotone. A function G is said to be *completely monotone* if it continuously differentiable to any order and

$$(-1)^n\,D^n\,G(t) \geq 0 \qquad \text{for all non-negative integers } n \text{ and } t > 0 \tag{7}$$

Bernstein's theorem [31, 15] asserts that eq. (3) is *equivalent* to (7). There is no such simple characterization of functions which have the form (2).

It was established in [3, 18, 13] that viscoelastic relaxation moduli are completely monotone.

For us the main benefit from using (3) instead of (2) is the equivalence of (3) with a property of the dispersion and attenuation that will be explained below. That is, certain statements about attenuation and dispersion functions follow from (3) or, conversely, imply that (3) does not hold.

We now recall the definition of a Bernstein function [2]. A differentiable function f on $]0, \infty[$ is a Bernstein function if $f \geq 0$ and its derivative f' is completely monotone. A Bernstein function is non-negative, continuous on $]0, \infty[$ and non-decreasing, hence it has a finite value at 0. A function f on $[0, \infty[$ which has the form $f(x) = x^2 \, \tilde{g}(x)$ for some Bernstein function g is called a *complete Bernstein function* (CBF) [15]. It can be proved that every complete Bernstein function is a Bernstein function [15].

The following facts about complete Bernstein functions will be needed here.

Theorem 2.1
A real function f on $]0, \infty[$ is a complete Bernstein function if (i) f has an analytic continuation $f(z)$ to the complex plane cut along the negative real axis; (ii) $f(0) \geq 0$, (iii) $f(\bar{z}) = \overline{f(z)}$, (iv) $\operatorname{Im} f(z) \geq 0$ in the upper half plane $\operatorname{Im} z \geq 0$.

Theorem 2.1 has the following corollaries: (1) If f is a CBF then f^α is a CBF if $0 \leq \alpha \leq 1$ [14].
(2) If $f \not\equiv 0$ then the function $x/f(x)$ is a CBF.

Every complete Bernstein function f has the integral representation:

$$f(x) = a + b\, x + x \int_{]0,\infty[} \frac{\nu(\mathrm{d}r)}{x + r} \tag{8}$$

where $a, b \geq 0$ and ν is a positive measure satisfying the inequality

$$\int_{[0,\infty[} \frac{\nu(\mathrm{d}r)}{1 + r} < \infty \tag{9}$$

[15].

If G is a completely monotone function integrable over $[0, 1]$ and satisfying eq. (3) then

$$\tilde{G}(p) = \int_{[0,\infty[} \frac{\mu(\mathrm{d}r)}{p + r} = \frac{\mu(\{0\})}{p} + \int_{]0,\infty[} \frac{\mu(\mathrm{d}r)}{p + r}$$

It follows that

$$\mathcal{Q}(p) := p\, \tilde{G}(p) = \mu(\{0\}) + \int_{]0,\infty[} \frac{\mu(\mathrm{d}r)}{p + r} \tag{10}$$

is a complete Bernstein function [14].

3   Dispersion and attenuation.

The Green's function $\mathcal{G}(t, x)$ is defined as the solution of the problem

$$\rho u_{,tt} = G(t) * u_{,txx} + \delta(x)\, \delta(t) \tag{11}$$

with zero initial data. In a three-dimensional space

$$\mathcal{G}(t,x) = \frac{-1}{(2\pi)^3 \, r} \int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} e^{pt} \, \frac{1}{\mathcal{Q}(p)} \, dp \int_{-\infty}^{\infty} \frac{e^{ikr}}{k^2 + B(p)^2} \, k \, dk =$$

$$\frac{1}{8\pi^2 \, i \, r} \int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} \frac{1}{\mathcal{Q}(p)} e^{p \, t - B(p) \, r} \, dp \quad (12)$$

where

$$B(p) := p \, \frac{\rho^{1/2}}{\mathcal{Q}(p)^{1/2}} \quad (13)$$

If G is completely monotone then $\mathcal{Q}$ is a complete Bernstein function. If a function f is a complete Bernstein function then the functions $f^{1/2}$ and $p/f(p)$ are complete Bernstein functions. Hence B is a complete Bernstein function and $B(p) = a + c\,p + b(p)$ where $a, c \geq 0$ and the dispersion-attenuation function b has the integral representation

$$b(p) := p \int_{]0,\infty[} \frac{\nu(dr)}{p+r} \quad (14)$$

and ν is a positive measure satisfying (9). A function b having an integral representation (14) will be called an *admissible dispersion-attenuation function*. The measure ν is the spectral measure of the dispersion-attenuation function b. A dispersion-attenuation function b of a viscoelastic material with a positive relaxation spectrum is admissible. Conversely, for any admissible dispersion-attenuation function there is a completely monotone relaxation modulus G satisfying eq. (13) and eq. (10).

Using this equivalence it is possible to decouple the study of admissible dispersion-attenuation functions from considering specific constitutive equations. Furthermore, in order to avoid incompatibility with viscoelastic theory experimental studies should target the spectral measure ν of b rather than the dispersion-attenuation function $b(p)$. Any positive measure ν satisfying (9) is compatible with the theory and the implications of the theory for the dispersion-attenuation function b are encapsulated in (14). Using the spectral measure ν instead of b to match the experimental data is analogous to expressing relaxation modulus in terms of the relaxation spectrum, e.g. by applying Prony sums to represent experimental data. A discretization of the dispersion-attenuation measure is presented in Sec. 5.1.

Let $G_\infty = \lim_{t\to\infty} G(t)$. Since $\lim_{p\to 0} \left[p \, \tilde{G}(p) \right] = G_\infty$, the inequality $G_\infty > 0$ and eq. (13) imply that $B(0) = 0$ and therefore $a = 0$. The limit $\lim_{p\to\infty} B(p)/p = c$ implies that $c = 1/c_0$, where c_0 denotes the wave front speed. The last conclusion follows from the fact that for $p > 1$

$$\frac{1}{p+r} \leq \frac{1}{1+r}$$

and the right-hand side is integrable with respect to the measure ν in view of eq. (9). For $p \to \infty$ the integrand of

$$\int_{]0,\infty[} \frac{\nu(\mathrm{d}r)}{p+r}$$

tends to zero, hence, by the Lebesgue Dominated Convergence Theorem, the integral tends to zero as well. It follows additionally that $b(p) = \mathrm{o}[p]$ for $p \to \infty$. Hence (12) can be recast in a more explicit form

$$\mathcal{G}(t,x) = \frac{1}{8\pi^2 \, \mathrm{i} \, r} \int_{-\mathrm{i}\infty+\varepsilon}^{\mathrm{i}\infty+\varepsilon} \frac{1}{\mathcal{Q}(p)} \mathrm{e}^{p\,(t-|x|/c_0)-b(p)\,r} \, \mathrm{d}p \tag{15}$$

and the dispersion-attenuation function $b(p)$ has sublinear growth.

We also note that the attenuation function is non-negative:

$$\mathcal{A}(p) := \mathrm{Re}\,b(p) = \int_{]0,\infty[} \frac{|p|^2 + r\,\mathrm{Re}\,p}{|p+r|^2} \nu(\mathrm{d}r) \geq 0 \tag{16}$$

for $\mathrm{Re}\,p \geq 0$.

The derivative

$$b'(p) = b(p)/p - \int_{]0,\infty[} \frac{\nu(\mathrm{d}r)}{(p+r)^2} \leq b(p)/p \tag{17}$$

Moreover

$$b'(p) = \int_{]0,\infty[} \frac{r\,\nu(\mathrm{d}r)}{(p+r)^2} \geq 0$$

Hence $b'(p) = \mathrm{o}[1]$ for $p \to \infty$.

The attenuation function $\mathcal{A}$ and the dispersion function $\mathcal{D}(p) := -\mathrm{Im}\,b(p)$ satisfy linear dispersion equations in parametric form:

$$\mathcal{A}(p) = \int_{]0,\infty[} \frac{|p|^2 + r\,\mathrm{Re}\,p}{|p+r|^2} \nu(\mathrm{d}r) \tag{18}$$

$$\mathcal{D}(p) = -\mathrm{Im}\,p \int_{]0,\infty[} \frac{r}{|p+r|^2} \nu(\mathrm{d}r) \tag{19}$$

The measure ν represents the dispersion-attenuation spectrum. An elementary dispersion-attenuation is represented by the function $p/(p+r)$ for a fixed value of r.

4   Implications of finite speed of wave propagation.

Since $1/\mathcal{Q}(p) = p\,\tilde{J}(p) = J(0) + \widetilde{J'}(p)$, where J is the creep compliance, we can express the Green's function (15) in the form of a convolution

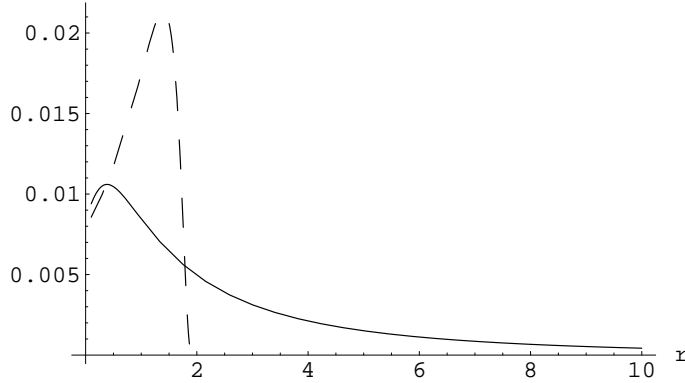$$\mathcal{G}(t,x) = J'(t) * H(t-|x|/c_0, |x|) + J_0\,H(t-|x|/c_0, |x|)$$

Figure 1: Two snapshots of Green's functions for power-law attenuation with $\alpha = 3/2$ (solid line) and $\alpha = 1/2$ (dashed line).

where $H(\tau, r)$ is the inverse Laplace transform of the function $e^{-b(p)\,r}/(4\pi r)$. In terms of the inverse Fourier transformation

$$H(\tau, r) = \frac{1}{8\pi^2\,r} \int_{-\infty}^{\infty} e^{-i\omega\tau - b(-i\omega)\,r}\,d\omega$$

Note that $b(-i\omega) = \overline{b(i\omega)}$. The integrand is square integrable if $r > 0$ and

$$\int_0^{\infty} e^{-2\,\mathrm{Re}\,b(-i\omega)\,r}\,d\omega < \infty \tag{20}$$

If eq. (20) holds then the Paley-Wiener theorem (Theorem XII in [19]) can be applied. By this theorem $H(\tau, r) = 0$ for $\tau < 0$ if and only if

$$\int_{-\infty}^{\infty} \frac{\mathrm{Re}\,b(-i\omega)}{1 + \omega^2}\,d\omega < \infty \tag{21}$$

J' is a causal function. Hence, if $H(\tau, r) = 0$ for $\tau < 0$ then

$$\mathcal{G}(t, x) = \int_0^{t-|x|/c_0} J'(s)\,H(t - |x|/c_0 - s, |x|)\,ds$$

vanishes for $t < |x|/c_0$.

In particular, if $b(p) \sim_{\infty} a\,p^{\alpha}$, $a > 0$, $\mathrm{Re}\,p \geq 0$, eqs (20) and (21) are ensured by the inequality $0 < \alpha < 1$. A precursor appears for $\alpha > 1$, as can be seen from Fig. 1. For $\alpha = 3/2$[1] there is no wavefront and the peak is preceded by a precursor extending to infinity. The limit case, as we already know, is the

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[1] For $\alpha = 1/3, 2/3$ the α-stable probability can be expressed in terms of Airy functions, see [12].

asymptotic behavior $b(p) \sim_\infty a\,p/\ln(p)$ and it entails unbounded propagation speed.

For a general viscoelastic medium with a positive relaxation spectrum we note that

$$\operatorname{Re} b(-i\omega) = \omega \operatorname{Im} \int_{]0,\infty[} \frac{\nu(d\xi)}{\xi - i\omega} = \omega^2 \int_{]0,\infty[} \frac{\nu(d\xi)}{\xi^2 + \omega^2} \qquad (22)$$

If the total mass M of ν is finite then

$$\lim_{\omega \to \infty} \operatorname{Re} b(-i\omega) = M \qquad (23)$$

by the Lebesgue Dominated Convergence Theorem. In the general case the inequality

$$\frac{\omega}{\xi^2 + \omega^2} \leq \frac{2\omega}{\xi^2 + 2\xi\omega + \omega^2} \leq \frac{2\omega}{(1 + \xi)(\xi + \omega)}$$

valid for $\omega \geq 1$ and the Lebesgue Dominated Convergence Theorem imply that $\operatorname{Re} b(-i\omega) = o[\omega]$ at $\omega \to \infty$.

We have thus proved that the asymptotic growth of $\operatorname{Re} b(-i\omega)$ in the high-frequency range is sublinear. This is however insufficient to ensure convergence of the left-hand side of (21) and vanishing of the wave field ahead of the wave-front $|x| = c_0\,t$. For example for $b(-i\omega) \sim_\infty \omega/\ln^\alpha(\omega)$ the integral in (21) does not always converge. Since we are interested in the high-frequency behavior, we can replace $1 + \omega^2$ by ω^2 in the denominator of (21). For $\alpha = 1$

$$\int_e^\omega \frac{dy}{y\,\ln(y)} = \ln(\ln(\omega))$$

is unbounded for $\omega \to \infty$, hence (21) is not satisfied. For $\alpha \neq 1$

$$\int_e^\omega \frac{dy}{y\,\ln^\alpha(y)} = \frac{\ln^{1-\alpha}(\omega) - 1}{1 - \alpha}$$

is unbounded for $\alpha < 1$ and bounded for $\alpha > 1$. We thus see that in contrast to the power law attenuation, linear growth is not a limit case for finite propagation speed. Some sublinear cases will also exhibit precursors ahead of the wave front.

In particular the function $b(p) = p/\ln^\alpha(1 + p)$ with $0 \leq \alpha \leq 1$ is a CBF with the asymptotic properties discussed in the previous paragraph. Indeed, for $\operatorname{Im} p \geq 0$ the argument $\psi = \arg(1 + p)$ satisfies the inequality $0 \leq \psi \leq \pi$. Hence $\ln(1 + p)$ maps the upper half plane into itself and is non-negative for $p \geq 0$. Hence, by Theorem 2.1, $\ln(1 + p)$ is a CBF. The same is true for $\ln^\alpha(1+p)$ if $0 < \alpha \leq 1$. In view of a property of CBF functions mentioned after Theorem 2.1 this implies that $b(p)$ is a CBF if $0 < \alpha \leq 1$. Moreover $b(0) = 0$ and $\lim_{p\to\infty} b(p)/p = 0$, hence $b(p)$ is an admissible dispersion-attenuation function. We thus have produced an example of an admissible dispersion-attenuation function $b(p)$ such that $b(-i\omega)$ has sublinear growth in the high frequency range but (21) is not satisfied.

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5 Examples.

5.1 Discrete dispersion-attenuation spectra.

If the measure $\nu = \sum_{n=1}^{N} c_n \, \varepsilon_{r_n}$, $r_n, c_n > 0$, where $\varepsilon_a = \delta(r-a)$ denotes the Dirac measure concentrated at the point a, then

$$b(p) = p \sum_{n=1}^{N} c_n/(p + r_n) \tag{24}$$

Eq. (24) can be used to construct numerical approximations of experimental data. Using the relation $\mathcal{Q}(p) \, B(p)^2 = \rho \, p^2$ and the assumption $\mu = \sum_{m=1}^{M} d_m \, \varepsilon_{s_m}$ with $d_m > 0$, $s_m \geq 0$ for $1 \leq m \leq M$, $s_m \neq r_n$ for all n, m, it is possible to express the relaxation data d_m, s_m in terms of the dispersion-attenuation data c_n, r_n and conversely.

5.2 Power-law attenuation.

Power-law attenuation is commonly used to match experimental dispersion and attenuation data for a wide variety of real viscoelastic materials such as polymers, bio-tissues and some viscoelastic fluids.

Consider the viscoelastic medium defined by the following equation of motion

$$\rho \left(D_C^2 + 2a \, D_C^{1+\alpha} + a^2 \, D_C^{2\alpha} \right) u = A \, \nabla^2 \, u + \delta(t) \, \delta(x) \tag{25}$$

with the initial data $u(0, x) = u_{,t}(0, x) = 0$ in d dimensions, $d = 1, 3$. D_C^α denotes the Caputo fractional derivative of order α

$$D_C^\alpha f(x) := \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f'(s) \, ds \tag{26}$$

[20].

It is assumed that $A > 0$, $a \geq 0$, $0 < \alpha < 1$. The Laplace transformation $\mathcal{L}_t$ with respect to the time variable and the Fourier transformation $\mathcal{F}_x$ with respect to the spatial variable bring eq. (25) to the following form

$$\rho \, g(p)^2 \, \hat{u}(p, \mathbf{k}) = -A \, \mathbf{k}^2 \, \hat{u}(p, \mathbf{k}) + 1 \tag{27}$$

where $\hat{u} := \mathcal{F}_x \left(\mathcal{L}_t(u) \right)$ and $g(p) := p + a \, p^\alpha$. Eq. (27) implies that in the power law attenuation model $\mathcal{Q}(p) = p \, A/g(p)^2$ and $B(p) = g(p)/c_0$.

We shall begin with solving eq. (25) in one dimension. Applying the inverse Fourier transformation to eq. (27):

$$\tilde{u}^{(1)}(p, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx+pt}}{\rho \, g(p)^2 + A \, k^2} \, dk$$

The contour can be closed by a large half-circle in upper-half complex k-plane if $x > 0$, in the lower- half complex k-plane if $x < 0$ and

$$\frac{1}{A \, k^2 + \rho \, g(p)^2} = \frac{1}{2ig(p) \, A} \left[\frac{1}{k - ig(p)/c_0} - \frac{1}{k + ig(p)/c_0} \right]$$

where $c_0 := \sqrt{A/\rho}$. We now restrict ourselves to imaginary values of $p = -iw$. Since $\mathrm{Im}[\mathrm{i}\, g(-iw)] = \mathrm{Re}\, g(-iw) = a\,|w|^\alpha\,\cos((1-\alpha)\,\pi/2) \geq 0$, the residuum at $\pm \mathrm{i}g(p)/c_0$ contributes if $\pm x > 0$. Hence

$$\tilde{u}(p, x) = \frac{1}{2A}\,\frac{\mathrm{e}^{-g(p)|x|/c_0 + pt}}{g(p)}$$

and

$$u^{(1)}(t, x) = \frac{1}{4\pi\mathrm{i}\,A}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{1}{g(p)}\mathrm{e}^{p\,t - g(p)|x|/c_0}\mathrm{d}p \qquad (28)$$

The solution $u^{(3)}$ of (25) in a three-dimensional space is given by the formula

$$u^{(3)}(t, x) = \frac{-1}{2\pi r}\frac{\partial}{\partial r}u^{(1)}(t, r)$$

Hence

$$u^{(3)}(t, x) = \frac{1}{8\pi^2\mathrm{i}|x|\,c_0\,A}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\mathrm{e}^{p\,(t-|x|/c_0) - a\,p^\alpha\,|x|}\mathrm{d}p =$$

$$\frac{1}{4\pi a^{1/\alpha}\,|x|^{1+1/\alpha}\,c_0\,A}\,P_\alpha\left((t-|x|/c_0)/(a|x|)^\alpha\right) \quad (29)$$

where

$$P_\alpha(z) := \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\mathrm{e}^{yz}\,\mathrm{e}^{-y^\alpha}\,\mathrm{d}y \qquad (30)$$

The function P_α is a totally skewed Lévy stable probability density [23, 32].

In the case of power-law attenuation $\mathrm{Re}\, b(-i\omega) = a\,\omega^\alpha\,\cos(\pi\alpha/2) > 0$ and for $p = -i\omega$

$$\left|\mathrm{e}^{p\,t - g(p)\,r/c_0}\right| = \mathrm{e}^{-a\,\omega^\alpha\,\cos(\alpha\,\pi/2)}$$

Hence ([30], Sec. 4.44) the integrals

$$f(t, r) := \frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{B}}(-1)^n\,p^{-2}\,p^m\,g(p)^{n+1}\,\mathrm{e}^{p\,t - g(p)\,r/c_0}\,\mathrm{d}p$$

are uniformly convergent for $r > 0$ and the derivatives $\partial^{n+m}f/\partial t^n\,\partial r^m$ exist for all positive integers n, m and $r > 0$.

All the properties of the dispersion-attenuation function b were derived from the fact that $B(p) = p\,\rho^{1/2}/\mathcal{Q}(p)^{1/2}$ is a complete Bernstein function. It follows from the above property of B that $\mathcal{Q}(p)^{1/2}$ is a complete Bernstein function [14]. It does not follow from here that $\mathcal{Q}$ is a also complete Bernstein function. Therefore G need not be a completely monotone function. A counterexample is provided by the power law attenuation model. If $0 \leq \alpha < 1/2$ then the relaxation modulus G is not completely monotone but the attenuation function satisfies eq. (14). The latter condition is satisfied for the power law attenuation if and only if $0 \leq \alpha < 1/2$ (Fig. 2).
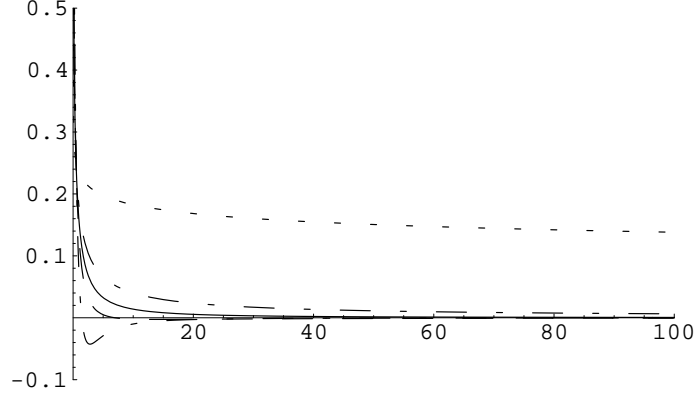
Figure 2: The relaxation modulus G for the power-law attenuation. The exponent values are $\alpha = 0.3, 0.4, 0.5, 0.6, 0.9$, from bottom to top.

Theorem 5.1

The spectral measure of the dispersion-attenuation function $a\,p^\alpha$, $0 \le \alpha < 1$, is

$$\nu(\mathrm{d}\xi) = \frac{\sin(\pi\alpha)}{\pi}\xi^{\alpha-1}\,\mathrm{d}\xi \tag{31}$$

Proof. The Laplace transforms

$$x^{\alpha-1} = \int_0^\infty \mathrm{e}^{-xy}\left[y^{-\alpha}/\Gamma(1-\alpha)\right]\,\mathrm{d}y \tag{32}$$

$$y^{-\alpha} = \int_0^\infty \mathrm{e}^{-yz}\left[z^{\alpha-1}/\Gamma(\alpha)\right]\,\mathrm{d}z \tag{33}$$

hence

$$x^{\alpha-1} = \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty \frac{z^{\alpha-1}}{x+x}\,\mathrm{d}z$$

which implies eq. (31).

$\square$

6 Superlinear power law attenuation.

A large number of papers have been devoted to the implications of the Kramers-Kronig dispersion relations for the wave number $K(\omega)$. The Kramers-Kronig dispersion relations would follow from the assumption that the function $K(\omega) - \omega/c_0$ is the Fourier transform of a causal function or causal tempered distribution $L(t)$. A priori the function $L(t)$ has no physical meaning and the assumption

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of causality of L is unwarranted. Causality of L would however be justified by the assumption that

$$B(p) = p/c_0 + \tilde{L}(p)$$

and the equation of motion has the following form

$$c_0^{-2} \, u_{,tt} + (2/c_0) \, L * u_{,t} + L * L * u = u_{,xx} \tag{34}$$

In this case $K(\omega) = \mathrm{i}B(-\mathrm{i}\omega) = \omega/c_0 + \mathrm{i}\,\tilde{L}(-\mathrm{i}\omega) = \mathrm{i}\omega/c_0 + \hat{L}(\omega)$. Eq. (34) is however incompatible with the viscoelastic constitutive equation. In a viscoelastic equation of motion integral operators should act on the Laplacian of u.

In [27] the authors try to guess the viscoelastic constitutive equation compatible with (34). Their approach involves an approximation of a spatial derivative by a temporal derivative. It is however possible to avoid an approximation by shifting the dispersive terms on the left-hand side of (34) to the right-hand side. The Laplace transform of the left-hand side of eq. (34) is

$$p^2 \left[1 + c_0 \, \tilde{L}(p)/p \right]^2 /c_0^2$$

assuming that $u(0,x) = u_{,t}(0,x) = 0$. Hence (34) has the form

$$c_0^{-2} \, u_{,tt} = [G(t) * u_{,tx}]_{,x}$$

where G is the inverse Laplace transform of

$$p^{-1} \left[1 + c_0 \, \tilde{L}(p)/p \right]^{-2} \tag{35}$$

Expression (35) is the Laplace transform of a completely monotone function if and only if $\left[1 + c_0 \, \tilde{L}(p)/p \right]^2 /p$ is the Laplace transform of a Bernstein function f [13, 24].

For $\tilde{L}(p) = a\,p^\alpha$, with $a, \alpha > 0$, the function f assumes the form $\theta(t) + 2c_0\,a\,t_+^{1-\alpha}/\Gamma(1-\alpha) + c_0^2\,a^2\,t_+^{2(1-\alpha)}/\Gamma(3-2\alpha)$. It is obvious that f is a Bernstein function (and G is completely monotone) if and only if $1/2 \le \alpha < 1$. We also note that $L(t) = a\,t_+^{-\alpha-1}/\Gamma(-\alpha)$ is a distribution. Convolution with $L(t)$ is a Riemann-Liouville fractional differential operator of order α. The order of the fractional differential equation (34) is 2 if $\alpha \le 1$ and 2α if $\alpha > 1$. (In the literature the highest-order derivative $L * L * u$ is often incorrectly neglected). For $\alpha > 1$ the order of the time-differential operator exceeds the order of the spatial differential operator, hence the equation is formally parabolic.

Concerning superlinear power law attenuation, note that the logarithmic attenuation rate

$$\mathcal{A}(\omega) := \mathrm{Re}\,B(-\mathrm{i}\omega) = a\,\cos(\alpha\,\pi/2)\,|\omega|^\alpha \tag{36}$$

is assumed non-negative, hence $1 < \alpha < 3$ must entail that $a < 0$. As the frequency tends to infinity the frequency-dependent phase speed $c(\omega)$, given by

the equation $1/c(\omega) := \mathrm{Re}[\mathrm{i}B(-\mathrm{i}\omega)] = 1/c_0 + a\,|\omega|^{\alpha-1}\,\sin(\alpha\,\pi/2)$ increases to its maximum value c_0 if $0 \le \alpha < 1$ ("abnormal dispersion"). For $1 \le \alpha < 2$ it increases from c_0 at zero frequency to infinity at a finite frequency $\omega_1 = [c_0\,|a|\,\sin(\alpha\,\pi/2)]^{1/(\alpha-1)}$ and changes sign. This behavior is clearly unphysical. For $2 < \alpha < 3$ phase speed decreases from c_0 at $\omega = 0$ to 0 at infinite frequency ("normal dispersion").

7 Models with variable attenuation exponent.

The limits on the dissipation-attenuation exponent α actually apply to the asymptotic value of $b(p)$ at infinity. Is it possible that the experimentally measured power law behavior applies to the middle frequency range?

Define the variable dissipation-attenuation exponent as the function

$$\alpha(p) = \ln(b(p))/\ln(p), \qquad p > 1 \tag{37}$$

so that $b(p) = p^{\alpha(p)}$. This definition has a major flaw: a singularity at $p = 1$. The exponent decreases to $-\infty$ at $p \to 1-$ and restarts from ∞ to decrease towards its asymptotic values.

The simplest examples of dispersion-attenuation functions with variable exponent are

$$b_1(p) = c\,p^\alpha/(a + p^\alpha), \qquad 0 \le \alpha < 1 \tag{38}$$

and

$$b_2(p) = c\,(1 + \tau\,p)^{\alpha-\beta}\,(\tau\,p)^\beta, \qquad 0 < \beta \le \alpha < 1 \tag{39}$$

b_1 is a complete Bernstein function because $p/b_1(p) \equiv p/c + a\,p^{1-\alpha}/c$ is obviously a complete Bernstein function. Moreover $b_1(0) = 0$ and $\lim_{p\to\infty} b_1(p)/p = 0$. Hence b_1 is an admissible dispersion-attenuation function.

In order to prove that b_2 is a complete Bernstein function we need Theorem 2.1. We now note that $\arg b_2(p) = \arg(1 + p)^{\alpha-\beta} + \arg p^\beta \le (\alpha - \beta)\arg p + \beta\arg p = \alpha\arg p$. For p in the upper half-plane $0 \le \arg p \le \pi$, hence $0 \le \arg b_2(p) \le \pi$ and b_2 is a complete Bernstein function. Since $b_2(0) = 0$ and $\lim_{p\to\infty} b_1(p)/p = 0$, b_2 is admissible as a dispersion-relaxation function. In the first case, $b = b_1$, the attenuation exponent $\alpha(p) \le 0$ and $\alpha(p) \to 0$ from below as $p \to \infty$. In the second case $b = b_2$ and $\alpha(p)$ increases from $\alpha(0) = \beta$ to $\alpha(\infty) = \alpha$. It is thus likely that the exponent assumes a larger value in the high frequency range. Thus a value of the exponent above 1 in the middle frequency range is not likely.

8 Conclusions.

The class of admissible dispersion and attenuation functions can be characterized by a class of Radon measures. The theory applies only to sublinear attenuation growth in the high frequency range.

Superlinear growth of the attenuation function in the high frequency range is incompatible with the assumption of positive relaxation spectrum underlying theoretical and experimental viscoelasticity. Superlinear growth of attenuation also implies that the phase speed is unbounded for high frequencies and the main signal is preceded by a precursor of infinite extent.

Explanation of superlinear frequency dependence of attenuation observed in many real materials remains a challenging task.

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